On some upper bounds on the fractional chromatic number of weighted graphs

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Abstract

Given a weighted graph $G_{\mathbf{x}}$, where $(x(v) : v \in V)$ is a non-negative, real-valued weight assigned to the vertices of G, let $B(G_{\mathbf{x}})$ be an upper bound on the fractional chromatic number of the weighted graph $G_{\mathbf{x}}$; so $\chi_f(G_{\mathbf{x}}) \leq B(G_{\mathbf{x}})$. To investigate the worst-case performance of the upper bound B, we study the graph invariant

$$\beta(G) \;=\; \sup_{\mathbf{x}\neq 0} \frac{B(G_{\mathbf{x}})}{\chi_f(G_{\mathbf{x}})}.$$

In recent work a particular upper bound resulting from the generalization of the greedy coloring algorithm was considered and the corresponding graph invariant was studied. In this work, we study some stronger upper bounds on the fractional chromatic number and the corresponding graph invariants. We derive some bounds for these graph invariants and obtain some explicit expressions for some families of graphs.

1. Introduction

Let $G = (V, E)$ be a simple, undirected graph on vertex set $V = \{v_1, \ldots, v_n\}$. Let $\{I_1, \ldots, I_L\}$ be the set of all independent sets of G, and let $A = [a_{ij}]$ be the $n \times L$ vertex-independent set incidence matrix of G. Thus, $a_{ij} = 1$ if $v_i \in I_j$ and $a_{ij} = 0$ if $v_i \notin I_j$. If $G_{\mathbf{x}}$ is a weighted graph, where $(x(v) : v \in V)$ is a non-negative, real-valued weight assigned to the vertices, the fractional chromatic number $\chi_f(G_{\mathbf{x}})$ of $G_{\mathbf{x}}$ is defined as [11] the value of the linear program: min $\mathbf{1}^T t$ subject to $At \geq \mathbf{x}, t \geq 0$. Equivalently, $\chi_f(G_{\mathbf{x}})$ is the smallest value of T such that each vertex v can be assigned a subset of $[0, T]$ of total length (or measure) $x(v)$, with adjacent vertices being assigned subintervals that are non-overlapping (except possibly at the endpoints of the subintervals). In general, the subset of $[0, T]$ assigned to a vertex need not be one continuous interval, but it needs to have total length $x(v)$.

We assume throughout that G is connected, since if G was disconnected we can just work with each connected component separately and so the results provided here

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still hold. Given a particular upper bound $B(G_{\mathbf{x}})$ on the fractional chromatic number $\chi_f(G_{\mathbf{x}})$, we investigate the graph invariant

$$\beta(G) \; = \; \sup_{\mathbf{x} \neq 0} \frac{B(G_{\mathbf{x}})}{\chi_f(G_{\mathbf{x}})}.$$

By scaling $\mathbf{x}$ appropriately, we see that $\beta(G)$ is the supremum of $B(G_{\mathbf{x}})$ over all $\mathbf{x}$ satisfying $\chi_f(G_{\mathbf{x}}) = 1$.

The problem of computing the fractional chromatic number of a graph is known to be NP-hard [7]. A special case of this problem where the graph is a line graph was studied in [8], [9]. The work [6] discusses a graph invariant associated with the performance of a *lower* bound on the fractional chromatic number. In recent work [4], a particular upper bound, which we denote here by $B_1(G_{\mathbf{x}})$, was studied, and the corresponding graph invariant $\beta_1(G)$ was shown to equal the induced star number of the graph. In this work, we consider some stronger upper bounds, denoted by $B_2(G_{\mathbf{x}})$ to $B_5(G_{\mathbf{x}})$ and study the corresponding graph invariants $\beta_i(G)$. The strongest of these upper bounds is $B_5(G_{\mathbf{x}})$, and we obtain an explicit expression for $\beta_5(G)$ for some families of graphs. These upper bounds all have the additional property that they can be efficient computed and can be utilized for resources estimation problems in distributed systems [10], [3], [5].

In the sequel, our notation is standard [1]. $\Gamma(v)$ denotes the set of vertices adjacent to G, and $d(v) = |\Gamma(v)|$ is the degree of v. $\Delta = \Delta(G)$ is the maximum degree of a vertex in G. For $A \subseteq V$, $x(A) := \sum_{v \in A} x(v)$.

2. Preliminaries

In this section we recall some results from the literature and prove that the stronger upper bounds investigated here are in fact upper bounds. In the next section we study the corresponding graph invariants.

Given a weighted graph $G_{\mathbf{x}}$, define

$$B_1(G_{\mathbf{x}}) := \max_{v \in V} \{x(v) + x(\Gamma(v))\}.$$

Proposition 1. *[10]For a weighted graph $G_{\mathbf{x}}$, we have the upper bound*

$$\chi_f(G_{\mathbf{x}}) \leq B_1(G_{\mathbf{x}}).$$

Definition 2. *The induced star number of a graph G is defined by*

$$\sigma(G) := \max_{v \in V(G)} \alpha(G[\Gamma(v)]),$$

where $G[V']$ denotes the subgraph of G induced by $V' \subseteq V$ and $\alpha(G)$ denotes the maximum size of an independent set of G. Thus, the induced star number of a graph is the number of leaf vertices r in the maximum sized star subgraph $K_{1,r}$ of the graph. Note that $\sigma(G)$ equals 0 or 1 iff G is the disjoint union of complete graphs.

For any graph G, define the graph invariant

$$\beta_1(G) := \sup_{\mathbf{x} \neq 0} \frac{B_1(G_{\mathbf{x}})}{\chi_f(G_{\mathbf{x}})}.$$

Since $B_1(G_{\mathbf{x}})$ is an upper bound, $\beta_1(G) \geq 1$. A recent result is the following:

Theorem 3. *[4] Given any graph G, $\beta_1(G)$ is exactly equal to the induced star number $\sigma(G)$ of the graph.*

We now present some upper bounds which are stronger than $B_1(G_{\mathbf{x}})$. Given a weighted graph $G_{\mathbf{x}}$ and any designated vertex $v_1 \in V$, define

$$B_2(G_{\mathbf{x}}) := \max\{x(v_1) + x(\Gamma(v_1)), \max_{i=2,\dots,n} \{x(v_i) + x(\Gamma(v_i)) - \min_{v_j \in \Gamma(v_i)} x(v_j)\}\}.$$

Proposition 4. *For a weighted graph $G_{\mathbf{x}}$ and a designated vertex $v_1 \in V$, we have the upper bound $\chi_f(G_{\mathbf{x}}) \leq B_2(G_{\mathbf{x}})$.*

Proof: By the equivalent definition of $\chi_f(G_{\mathbf{x}})$ given above, it suffices to show that it is possible to assign a subset of $[0, B_2(G_{\mathbf{x}})]$ to each vertex such that the total length of intervals assigned to each $v \in V$ is $x(v)$ and adjacent vertices are assigned nonoverlapping subsets. Given $G_{\mathbf{x}}$ and v_1, order the remaining vertices as follows. Let v_2 be any vertex in G adjacent to v_1, let v_3 be any vertex adjacent to v_1 or v_2. Given $v_1, \dots, v_r$, let v_{r+1} be any vertex adjacent to one of the previous vertices. Now assign subsets of $[0, B_2(G_{\mathbf{x}})]$ to the vertices in reverse order. Assign v_n the subset $[0, x(v_n)]$. Once the vertices $v_n, v_{n-1}, \dots, v_{r+1}$ have been assigned subsets, by the definition of $B_2(G_{\mathbf{x}})$, v_r can also be assigned some subset of length $x(v_r)$ because v_r has at least one neighbor in $\{v_1, \dots, v_{r-1}\}$ which has not yet been assigned a subset. Finally, v_1 can also be assigned some subset because $B_2(G_{\mathbf{x}}) \geq x(v_1) + x(\Gamma(v_1))$. ∎

Given a weighted graph $G_{\mathbf{x}}$, define

$$B_3(G_{\mathbf{x}}) := \max_{v \in V}\{x(v) + x(\Gamma(v)) - \min_{w \in \Gamma(v)} x(w)\}.$$

Proposition 5. *Given $G_{\mathbf{x}}$, suppose G is not a complete graph or an odd cycle. Then $\chi_f(G_{\mathbf{x}}) \leq B_3(G_{\mathbf{x}})$.*

Proof: Let $r \geq 1$ be the minimum number of vertices whose removal disconnects G. Consider three cases, depending on the value of r (this proof method is from [2]).

$r = 1$: Let v_1 be a cutvertex of G and suppose that the removal of v_1 disconnects G into connected components $G_1, \dots, G_s$. Since each G_i is connected to v_1, by the definition of $B_3(G_{\mathbf{x}})$ and using v_1 as the designated vertex, each vertex of G_i can be assigned a subset of $[0, B_3(G_{\mathbf{x}})]$ using the method given in the proof of the previous proposition. Finally, v_1 can also be assigned some subset as follows. Let v_a and v_b be the neighbors of v_1 in components G_1 and G_2, respectively. Without loss of generality, assume $x(v_a) \leq x(v_b)$. Then the interval $[0, B_3(G_{\mathbf{x}})]$ and the corresponding subsets assigned to the vertices of G_1 can be permuted so that the subset assigned to v_a is a subset of the subset assigned to v_b. Since two neighbors of v_1 have been assigned

overlapping subsets, by the definition of $B_3(G_{\mathbf{x}})$, v_1 can also be assigned some subset of $[0, B_3(G_{\mathbf{x}})]$.

$r \geq 3$: Since G is not complete, there exist v_1, v_2 and v_3 such that v_1 is adjacent to v_2 and v_3 but v_2 and v_3 are nonadjacent. Assign v_2 the subset $[0, x(v_2)]$ and v_3 the subset $[0, x(v_3)]$. Since $r \geq 3$, the induced subgraph $G - \{v_2, v_3\}$, which contains a designated vertex v_1, is connected. Hence the proof method above can be applied, using v_1 as the designated vertex, to assign subsets of $[0, B_3(G_{\mathbf{x}})]$ to the vertices $v_4, \ldots, v_n$ in some order. Finally, v_1 can also be assigned a subset because two of its neighbors were assigned overlapping subsets.

$r = 2$: Let Δ denote the maximum degree of a vertex of G. If $\Delta \leq 2$, then G is an even cycle (since we assumed G is not an odd cycle). If i is even, assign the subset $[0, x(v_i)]$ to v_i, and if i is odd assign the subset $[B_3(G_{\mathbf{x}}) - x(v_i), B_3(G_{\mathbf{x}})]$ to v_i. This assignment satisfies the condition that adjacent vertices are assigned nonoverlapping subsets. Now assume $\Delta \geq 3$. There exist v_1 and v_2 such that v_1 is a cutvertex of $G - v_2$ and $G - \{v_1, v_2\}$ has connected components $G_1, \ldots, G_s$, with $s \geq 2$. Then, v_1 has neighbors v_a and v_b in G_1 and G_2, respectively. As before, subsets $[0, x(v_a)]$ and $[0, x(v_b)]$ can be assigned to v_a and v_b, respectively, and subsets can then be assigned to the remaining vertices using v_1 as the designated vertex, and finally a subset can be assigned to v_1 as well. ∎

Bounds $B_4(G_{\mathbf{x}})$ and $B_5(G_{\mathbf{x}})$ were proven to be upper bounds in [10] in the context of resource allocation in networks. For the sake of completeness, we give the short proofs here for these two bounds using the notation and terminology of fractional chromatic number.

Given a weighted graph $G_{\mathbf{x}}$, define

$$B_4(G_{\mathbf{x}}) := \max_{v \in V} \{x(v)[d(v) + 1]\}.$$

Proposition 6. *For any weighted graph $G_{\mathbf{x}}$, we have the upper bound $\chi_f(G_{\mathbf{x}}) \leq B_4(G_{\mathbf{x}})$.*

Proof: Given $G_{\mathbf{x}}$, order the vertices so that $x(v_1) \leq x(v_2) \leq \ldots \leq x(v_n)$. Assign v_1 the subset $[0, x(v_1)]$. Assume that vertices $v_1, \ldots, v_r$ have already been assigned subsets. By the inequality $x(v_{r+1})[d(v_{r+1}) + 1] \leq B_4(G_{\mathbf{x}})$ and by the chosen ordering of the vertices, it follows that

$$x(v_{r+1}) + x(\Gamma(v_{r+1}) \cap \{v_1, \ldots, v_r\}) \leq B_4(G_{\mathbf{x}}).$$

Hence, it is possible to assign to v_{r+1} some subset of $[0, B_4(G_{\mathbf{x}})]$ that is nonoverlapping with the subsets assigned to its neighbors in $\{v_1, \ldots, v_r\}$. It follows by induction that $\chi_f(G_{\mathbf{x}}) \leq B_4(G_{\mathbf{x}})$. ∎

We can combine the bounds B_1 and B_4 to get a strictly better bound. For a weighted graph $G_{\mathbf{x}}$, define

$$B_5(G_{\mathbf{x}}) := \max_{v \in V} \min\{x(v) + x(\Gamma(v)), x(v)[d(v) + 1]\}.$$

Proposition 7. *Given a weighted graph $G_{\mathbf{x}}$, we have the upper bound $\chi_f(G_{\mathbf{x}}) \leq B_5(G_{\mathbf{x}})$.*

Proof: Given $G_{\mathbf{x}}$, order the vertices of G so that $x(v_1) \leq x(v_2) \leq \ldots, \leq x(v_n)$. Assign v_1 the subset $[0, x(v_1)]$. Assume $v_1, \ldots, v_r$ have already been assigned subsets. By the definition of $B_5(G_{\mathbf{x}})$, either $x(v_{r+1}) + x(\Gamma(v_{r+1})) \leq B_5(G_{\mathbf{x}})$ or $x(v_{r+1})[d(v_{r+1}) + 1] \leq B_5(G_{\mathbf{x}})$. In either case, due to the chosen ordering of the vertices, we have that $x(v_{r+1}) + x(\Gamma(v_{r+1}) \cap \{v_1, \ldots, v_r\}) \leq B_5(G_{\mathbf{x}})$. Thus, it possible to assign some subset of $[0, B_5(G_{\mathbf{x}})]$ to v_{r+1}. The assertion follows by induction. ∎

3. Main results

When G is not a complete graph or an odd cycle, the upper bound $B_3(G_{\mathbf{x}})$ holds, and we then have the following result.

Define the graph invariant

$$\beta_3(G) := \sup_{\mathbf{x} \neq 0} \frac{B_3(G_{\mathbf{x}})}{\chi_f(G_{\mathbf{x}})}.$$

Theorem 8. *Suppose G is not a complete graph or an odd cycle. Let $S := \{s \in V : \alpha(G[\Gamma(s)]) = \sigma(G)\}$ denote the set of vertices of G that can induce a maximum size star with some of their neighbors. Then, $\beta_3(G)$ equals $\sigma(G) - 1$ if every vertex in S has degree $\sigma(G)$, and $\beta_3(G)$ equals $\sigma(G)$ otherwise.*

Proof: For brevity let σ denote $\sigma(G)$, and let v_0 be a vertex of G whose neighbors $v_1, \ldots, v_\sigma$ form an independent set. Pick $\mathbf{x}$ to be a 0-1 vector as follows: $x(v) = 1$ if $v \in \{v_1, \ldots, v_\sigma\}$, and $x(v) = 0$ otherwise. Then, $B_3(G_{\mathbf{x}}) = \sigma - 1$ and $\chi_f(G_{\mathbf{x}}) = 1$, so that $\beta_3(G) \geq \sigma(G) - 1$. Since $B_3(G_{\mathbf{x}}) \leq B_1(G_{\mathbf{x}})$, we have that $\beta_3(G) \leq \beta_1(G) = \sigma(G)$. Thus, $\sigma - 1 \leq \beta_3(G) \leq \sigma$.

Now let $S \subseteq V$ be as defined in the assertion. Pick any weight $\mathbf{x}$, and assume without loss of generality that $\chi_f(G_{\mathbf{x}}) = 1$. Recall that $\chi_f(G_{\mathbf{x}})$ is the value of the linear program: min $\mathbf{1}^T t$ subject to $At \geq x, t \geq 0$. An optimal solution to this program gives an assignment of subsets of $[0, \chi_f(G_{\mathbf{x}})]$ to each vertex v such that the union of subsets assigned to $\Gamma(v)$ is nonoverlapping with the subset assigned to v. Hence, the subset assigned to any $w \in \Gamma(v)$ has length at most $1 - x(v)$. We consider the two cases given in the assertion.

(i) Suppose that all vertices of S have degree σ, and let $v \in V$. First assume $v \in S$. Then,

$$x(v) + \{x(\Gamma(v)) - \min_{w \in \Gamma(v)} x(w)\} \leq x(v) + (\sigma - 1)(1 - x(v)) = \sigma - 1 + x(v)(2 - \sigma),$$

which is at most $\sigma - 1$ since $2 - \sigma \leq 0$ when G is not complete. Now assume $v \notin S$. Then, $\alpha(G[\Gamma(v)]) \leq \sigma - 1$. Let ϵ denote the nonnegative quantity $\min_{w \in \Gamma(v)} x(w)$. Then

$$B_3(G_{\mathbf{x}}) = x(v) + x(\Gamma(v)) - \epsilon \leq x(v) + (\sigma - 1)(1 - x(v)) - \epsilon \leq \sigma - 1 + x(v)(2 - \sigma),$$

which is at most $\sigma - 1$ since $\sigma \geq 2$. Hence, for all $v \in V$ and any weight $\mathbf{x}$ satisfying $\chi_f(G_\mathbf{x}) = 1$, we have that $B_3(G_\mathbf{x}) \leq \sigma - 1$. This establishes that $\beta_3(G) = \sigma - 1$ if all vertices of S have degree σ.

(ii) Now suppose that some vertex $v_0 \in S$ has degree at least $\sigma+1$. Let $v_1, \ldots, v_{\sigma+1}$ be the neighbors of v_0 such that $v_1, \ldots, v_\sigma$ form an independent set. Pick $\mathbf{x}$ as follows: $x(v) = 1$ if $v \in \{v_1, \ldots, v_\sigma\}$, and $x(v) = 0$ otherwise. Then $x(v_0) + x(\Gamma(v_0)) - \min_{w \in \Gamma(v_0)} x(w) = \sigma$. Hence, $B_3(G_\mathbf{x}) \geq \sigma$. Also, $\chi_f(G_\mathbf{x}) = 1$. Thus, $\beta_3(G) \geq \sigma$. The opposite inequality $\beta_3(G) \leq \sigma$ is already known. Hence, $\beta_3(G) = \sigma$ in this case. ∎

Given a graph G, define the graph invariant

$$\beta_4(G) := \sup_{\mathbf{x} \neq 0} \frac{B_4(G_\mathbf{x})}{\chi_f(G_\mathbf{x})}.$$

Lemma 9. *For any graph G, we have that $\beta_4(G) = \Delta(G) + 1$.*

Proof: Let v_0 be a vertex of degree Δ having neighbors $v_1, \ldots, v_\Delta$. Pick $\mathbf{x}$ as follows: $x(v) = 1$ if $v = v_0$, and $x(v) = 0$ otherwise. Then $x(v_0)[d(v_0) + 1] = \Delta + 1$, so that $\beta_4(G) \geq \Delta + 1$. To prove the opposite inequality, pick any $\mathbf{x} \neq 0$. Let $v \in V$. We know that $x(v) \leq \chi_f(G_\mathbf{x})$. Hence, $x(v)[d(v) + 1] \leq \chi_f(G_\mathbf{x})(\Delta + 1)$. It follows that $\beta_4(G) \leq \Delta + 1$. ∎

For any graph G, define the graph invariant

$$\beta_5(G) := \sup_{\mathbf{x} \neq 0} \frac{B_5(G_\mathbf{x})}{\chi_f(G_\mathbf{x})}.$$

Theorem 10. *For any graph G, the graph invariant $\beta_5(G)$ satisfies the bounds*

$$\frac{\sigma(G) + 1}{2} \leq \beta_5(G) \leq \sigma(G).$$

Moreover, the lower and upper bounds are tight; the star graphs realize the lower bound, and there exist graph sequences for which $\beta_5(G)$ approaches the upper bound arbitrarily closely.

Proof: Since $B_5(G_\mathbf{x}) \leq B_1(G_\mathbf{x})$, we have the upper bound $\beta_5(G) \leq \beta_1(G) = \sigma(G)$. To prove the remaining parts of the theorem, we first determine $\beta_5(G)$ for a family of graphs that includes the star graphs as a special case. The property that any member G of the family needs to satisfy is that G has some vertex $u \in V$ that is adjacent to all the other vertices of G and whose removal disconnects G into a disjoint union of complete graphs. i.e., u has degree $|V| - 1$ and $\sigma(G - u) \leq 1$.

We now claim the following: Suppose G has a vertex u of degree $|V| - 1$ and the removal of u produces disjoint complete graphs on vertex sets $V_1, \ldots, V_\eta$. Then

$$\beta_5(G) = \frac{\eta(1 + \sum |V_i|)}{\eta + \sum |V_i|}.$$

To prove this claim, recall that $\beta_5(G)$ is the supremum of

$$\max_{v \in V} \min\{x(V) + x(\Gamma(v)), x(v)[d(v) + 1]\}$$

over all $\mathbf{x}$ satisfying $\chi_f(G_{\mathbf{x}}) = 1$. Let $\delta := x(u)$. Then, $\chi_f(G_{\mathbf{x}}) = 1$ implies that $x(V_i) \leq 1 - \delta$, for $i = 1, \ldots, \eta$. Hence, for any $w \in V_i$, $x(w) + x(\Gamma(w)) \leq 1$. But $\beta_5(G) \geq 1$ since $B_5(G_{\mathbf{x}})$ is an upper bound. Hence, the maximum above is attained at the vertex u. So, $\beta_5(G)$ is equal to

$$\sup_{\delta\in[0,1]} \min\{\delta + (1-\delta)\eta, \delta(1 + \sum |V_i|)\}.$$

It can be verified that $\delta + (1-\delta)\eta \leq \delta(1 + \sum |V_i|)$ iff $\delta \geq \frac{\eta}{\eta+\sum|V_i|}$ and that $\beta_5(G)$ attains its optimal value of $\frac{\eta(1+\sum|V_i|)}{\eta+\sum|V_i|}$ when $\delta = \frac{\eta}{\eta+\sum|V_i|}$. This proves the claim.

In the special case that each $|V_i| = 1$, G is the star graph $K_{1,\eta}$ and $\beta_5(G)$ evaluates to $\frac{1+\eta}{2}$. This proves the lower bound since every graph G has a star $K_{1,\sigma}$ as an induced subgraph. We have also shown that the class of star graphs realize this lower bound.

In the special case where $|V_1|$ approaches infinity, $\frac{\eta(1+\sum|V_i|)}{\eta+\sum|V_i|}$ approaches η, which equals $\sigma(G)$. Hence, the upper bound in the assertion is tight. ∎

In the previous proof, the exact value of $\beta_5(G)$ was determined if G had a vertex x of degree $|V| - 1$ satisfying the condition $\sigma(G - x) \leq 1$. While the star graphs and complete graphs satisfy this condition, the even and odd cycles and bipartite graphs do not. One general class of graphs that includes the family of star graphs, the complete graphs, the cycles, and the bipartite graphs are those that satisfy the following property: for each vertex $v \in V$, the neighbors of v induce a disjoint union of complete graphs. For this general class of graphs, we obtain an explicit expression for the exact value of $\beta_5(G)$.

Theorem 11. *Suppose that G satisfies the condition $\sigma(G[\Gamma(v)]) \leq 1$ for each $v \in V$. Let $\eta(v)$ denote the number of connected components induced by the neighbors of v. Then,*

$$\beta_5(G) = \max_{v\in V} \frac{\eta(v)[1 + d(v)]}{\eta(v) + d(v)}.$$

Proof: Recall that $\beta_5(G)$ is the supremum of

$$\max_{v\in V} \min\{x(v) + x(\Gamma(v)), x(v)[d(v) + 1]\}$$

taken over all $\mathbf{x}$ satisfying $\chi_f(G_{\mathbf{x}}) = 1$. Fix any weight $\mathbf{x}$ and pick a vertex $u \in V$. Since $\sigma(G[\Gamma(u)]) \leq 1$, $\Gamma(u)$ induces a disjoint union of complete graphs on vertex sets $U_1, \ldots, U_\eta$, say. From the previous proof, we know that for this $\mathbf{x}$,

$$\min\{x(u) + x(\Gamma(u)), x(u)[d(u) + 1]\} \leq \frac{\eta(1 + \sum|U_i|)}{\eta + \sum|U_i|} = \frac{\eta(1 + d(u))}{\eta + d(u)}.$$

Hence,

$$\beta_5(G) \leq \sup_{\mathbf{x}\neq 0} \max_{u\in V} \frac{\eta(1 + d(u))}{\eta + d(u)} = \max_{u\in V} \frac{\eta(1 + d(u))}{\eta + d(u)}.$$

To prove the opposite inequality, define

$$u^* := \arg\max_{u\in V} \frac{\eta(1 + d(u))}{\eta + d(u)}.$$

Suppose $\Gamma(u^*)$ induces disjoint complete graphs on vertex sets $U_1^*, \ldots, U_s^*$. Now pick $\mathbf{x}$ as follows. Let

$$x(u) = \frac{\eta(u^*)}{\eta(u^*) + d(u^*)}$$

if $u = u^*$, and let

$$x(u) = \frac{1 - x(u^*)}{|U_i^*|}$$

if $u \in U_i^*$, and let $x(u) = 0$ for the remaining vertices. For this choice of $\mathbf{x}$, $\chi_f(G_{\mathbf{x}}) = 1$, and the maximum

$$B_5(G_{\mathbf{x}}) = \max_{u \in V} \min\{x(u) + x(\Gamma(u), x(u)[d(u) + 1]\}$$

is attained at $u = u^*$, and this maximum equals $\frac{\eta(u^*)[1+d(u^*)]}{\eta(u^*)+d(u^*)}$. Hence, $\beta_5(G)$ is at least this quantity. Hence, $\beta_5(G)$ equals this quantity. ∎

Corollary 12. *If G is a star graph, a bipartite graph or a cycle, then*

$$\beta_5(G) = \frac{1 + \Delta(G)}{2}.$$

Proof: Observe that if G is a star graph, a bipartite graph or a cycle, and v is a vertex of G, then $\eta(v) = d(v)$, so that

$$\frac{\eta(v)[1 + d(v)]}{\eta(v) + d(v)} = \frac{1 + d(v)}{2}.$$

∎

The simplest example of a graph that does not satisfy the conditions of Theorem 11 is $K_4 - e$. For this graph, a straightforward but lengthy computation yields the exact value of the graph invariant to be $\beta_5(K_4 - e) = 1.6$.

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